Boundary K-matrices for the XYZ, XXZ and XXX spin chains

1994 J. Phys. A: Math. Gen. 27 6129

(http://iopscience.iop.org/0305-4470/27/18/021)

J. Phys. A: Math. Gen. 27 (1994) 6129–6137. Printed in the UK

Boundary K-matrices for the XYZ, XXZ and XXX spin chains

H J de Vega and A González-Ruiz†
LPTHE, Tour 16, 1er étage, Université Paris VI, 4 Place Jussieu, 75252 Paris cedex 05, France

Received 5 April 1994

Abstract. The general solutions for the factorization equations of the reflection matrices $K^{\pm}(\theta)$ for the eight-vertex and six-vertex models (XYZ, XXZ and XXX chains) are found. The associated integrable magnetic Hamiltonians are derived explicitly, finding families depending on several continuous as well as discrete parameters.

1. Introduction

It is clearly interesting to find the widest possible class of boundary conditions (BC) compatible with integrability associated with a given model.

Not all BC obey this requirement. Periodic and twisted (under a symmetry of the model) BC are usually compatible with the Yang–Baxter equations [1, 2]. In addition, there are the BC defined by reflection matrices $K^{\pm}$ [3–6]. These $K^{\pm}$ matrices can be interpreted as defining the scattering by the boundaries. In a recent publication [7], the interpretation of these matrices as boundary S-matrices in two-dimensional integrable quantum field theories was developed. They also imply boundary terms for the spin Hamiltonians which can be interpreted as the coupling with magnetic fields on the edges of the chain.

In addition, quantum group invariance arises for specific choices of fixed BC (see for example [5, 8, 9] for the trigonometric case and [10] for the elliptic case). A quantum-group-like structure is still to be found for which Baxter's eight-vertex elliptic matrix [1] could act as an intertwiner (for a recent attempt see [11]) giving an affine quantum invariance to the infinite spin chain and the boundary terms for the quantum group invariance of the finite chain. This program has been achieved in the elliptic case for the free-fermionic model (see [10, 12]).

A general setting for finding boundary terms compatible with integrability was proposed by Sklyanin [3]. To find these BC, one has to solve the so-called reflection equations:

$$R(\theta-\theta')[K^-(\theta)\otimes 1]R(\theta+\theta')[K^-(\theta')\otimes 1] = [K^-(\theta')\otimes 1]R(\theta+\theta')[K^-(\theta)\otimes 1]R(\theta-\theta') \tag{1}$$

$$R(\theta-\theta')[1\otimes K^+(\theta)]R(\theta+\theta')[1\otimes K^+(\theta')] = [1\otimes K^+(\theta')]R(\theta+\theta')[1\otimes K^+(\theta)]R(\theta-\theta') \tag{2}$$

where $R(\theta)$ is the R-matrix of the chain and $K^{\pm}(\theta)$ give the boundary terms (see below).

† Permanent address: Departamento de Física Teórica, Universidad Complutense, 28040 Madrid, Spain.

As is known, the XYZ model is obtained from the elliptic eight-vertex solution of the Yang–Baxter equation:

$$[1\otimes R(\theta-\theta')][R(\theta)\otimes 1][1\otimes R(\theta')] = [R(\theta')\otimes 1][1\otimes R(\theta)][R(\theta-\theta')\otimes 1]. \tag{3}$$

The XXZ and XXX models follow, respectively, from the trigonometric and rational limits of this R-matrix.

We present in this paper the general solutions $K^{\pm}(\theta)$ to these equations for the XYZ, XXZ and XXX models. We find, for the elliptic case, two families of solutions, each family depending on one continuous and one discrete parameter; see equations (35) and (36). For the trigonometric and rational limit, we find a family of solutions depending on four continuous parameters; see equations (44) and (52), respectively.

We remark that the trigonometric limit of the elliptic solutions of (1) and (2) does not provide all the solutions to the trigonometric/hyperbolic case.

From these $K^{\pm}(\theta)$ solutions, we derive the boundary terms in the XYZ Hamiltonian which are compatible with integrability. Finally, we analyse the relation of the present eight-vertex results with the general K-matrices of the six-vertex results reported in [4] and consider, in addition, the rational limit.

2. General solution to the reflection equations for the eight-vertex model (XYZ chain)

The R-matrix for the XYZ chain can be written as [1]

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & k\,\mathrm{sn}\,\gamma\,\mathrm{sn}\,\theta \\ 0 & \dfrac{\mathrm{sn}\,\gamma}{\mathrm{sn}(\theta+\gamma)} & \dfrac{\mathrm{sn}\,\theta}{\mathrm{sn}(\theta+\gamma)} & 0 \\ 0 & \dfrac{\mathrm{sn}\,\theta}{\mathrm{sn}(\theta+\gamma)} & \dfrac{\mathrm{sn}\,\gamma}{\mathrm{sn}(\theta+\gamma)} & 0 \\ k\,\mathrm{sn}\,\gamma\,\mathrm{sn}\,\theta & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

where sn (and cn, dn in the formulae below) stand for Jacobi elliptic functions of modulus $0 \leqslant k \leqslant 1$.

This solution of the Yang–Baxter equations enjoys the following properties:

(i) regularity: $R(0) = 1$;

(ii) parity invariance: $PR(\theta)P = R(\theta)$ where $P^{ab}_{cd} = \delta^a_d\delta^b_c$;

(iii) time-reversal invariance: $R^{ab}_{cd} = R^{cd}_{ab}$ and;

(iv) crossing unitarity: $\hat{R}(\theta)\hat{R}(-\theta-2\eta) = \hat{\rho}(\theta)1$.

Here, $\hat{R}^{ab}_{cd} = R^{ac}_{bd}$, $\eta = \gamma$ and

$$\begin{aligned}\hat{\rho}(\theta) &= 1 - \frac{\mathrm{sn}^2\gamma}{\mathrm{sn}^2(\gamma+\theta)} \\ &= \frac{\mathrm{sn}(\theta+2\gamma)\,\mathrm{sn}\,\theta}{\mathrm{sn}^2(\gamma+\theta)}[1-k^2\,\mathrm{sn}^2\gamma\,\mathrm{sn}^2(\gamma+\theta)].\end{aligned} \tag{5}$$

From (3) and (i), unitarity follows

$$R(\theta)R(-\theta) = \rho(\theta)1 \tag{6}$$

$$\begin{aligned}\rho(\theta) &= 1 - k^2\,\mathrm{sn}^2\gamma\,\mathrm{sn}^2\theta \\ &= \frac{\mathrm{sn}^2\gamma - \mathrm{sn}^2\theta}{\mathrm{sn}(\gamma+\theta)\,\mathrm{sn}(\gamma-\theta)}.\end{aligned} \tag{7}$$

It is shown in [3] that when the R-matrix enjoys properties (ii)–(iv) and (6), we can look for solutions to equations (1) and (2) in order to find open BC compatible with integrability.

Since (ii) holds, equations (1) and (2) are equivalent. We now look for the general solution of these equations in the form

$$K(\theta) = \begin{pmatrix} x(\theta) & y(\theta) \\ z(\theta) & v(\theta) \end{pmatrix}. \tag{8}$$

Inserting equations (4) and (8) into (1), we find twelve independent equations:

$$\mathrm{b}^+ yz' + \mathrm{c}^+\mathrm{d}^- zz' = \mathrm{c}^+\mathrm{d}^- yy' + \mathrm{b}^+ zy' \tag{9}$$

$$\mathrm{d}^- vv' + \mathrm{d}^+ xv' = \mathrm{d}^+ vx' + \mathrm{d}^- xx' \tag{10}$$

$$\mathrm{b}^- yz' + \mathrm{c}^-\mathrm{d}^+ zz' = \mathrm{c}^-\mathrm{d}^+ yy' + \mathrm{b}^- zy' \tag{11}$$

$$\mathrm{b}^+\mathrm{c}^- vv' + \mathrm{b}^-\mathrm{c}^+ xv' = \mathrm{c}^+\mathrm{b}^- vx' + \mathrm{c}^-\mathrm{b}^+ xx' \tag{12}$$

$$\mathrm{c}^+ yx' + \mathrm{b}^+\mathrm{d}^- zx' + \mathrm{d}^- vz' + \mathrm{d}^+ xz' = \mathrm{c}^- yx' + \mathrm{b}^+\mathrm{c}^- vy' + \mathrm{b}^-\mathrm{c}^+ xy' + \mathrm{b}^-\mathrm{d}^+ zx' \tag{13}$$

$$\mathrm{b}^+ yv' + \mathrm{d}^-\mathrm{d}^+ vy' + xy' + \mathrm{c}^+\mathrm{d}^- zv' = \mathrm{b}^- yx' + \mathrm{b}^-\mathrm{b}^+ vy' + \mathrm{c}^-\mathrm{c}^+ xy' + \mathrm{c}^-\mathrm{d}^+ zx' \tag{14}$$

$$\mathrm{b}^-\mathrm{d}^+ yx' + \mathrm{c}^- zx' + \mathrm{b}^+\mathrm{c}^- vz' + \mathrm{b}^-\mathrm{c}^+ xz' = \mathrm{b}^+\mathrm{d}^- yx' + \mathrm{d}^- vy' + \mathrm{d}^+ xy' + \mathrm{c}^+ zx' \tag{15}$$

$$\mathrm{b}^- yv' + \mathrm{c}^-\mathrm{c}^+ vy' + \mathrm{b}^-\mathrm{b}^+ xy' + \mathrm{c}^-\mathrm{d}^+ zv' = \mathrm{b}^+ yx' + vy' + \mathrm{d}^-\mathrm{d}^+ xy' + \mathrm{c}^+\mathrm{d}^- zx' \tag{16}$$

$$\mathrm{c}^-\mathrm{d}^+ yx' + \mathrm{b}^- zx' + \mathrm{b}^-\mathrm{b}^+ vz' + \mathrm{c}^-\mathrm{c}^+ xz' = \mathrm{c}^+\mathrm{d}^- yv' + \mathrm{b}^+ zv' + z'x + \mathrm{d}^+\mathrm{d}^- vz' \tag{17}$$

$$\mathrm{c}^- yv' + \mathrm{b}^-\mathrm{c}^+ vy' + \mathrm{b}^+\mathrm{c}^- xy' + \mathrm{b}^-\mathrm{d}^+ zv' = \mathrm{c}^+ yv' + \mathrm{b}^+\mathrm{d}^- zv' + \mathrm{d}^+ vz' + \mathrm{d}^- xz' \tag{18}$$

$$\mathrm{c}^+\mathrm{d}^- yx' + \mathrm{b}^+ zx' + vz' + \mathrm{d}^-\mathrm{d}^+ xz' = \mathrm{c}^-\mathrm{d}^+ yv' + \mathrm{b}^- zv' + \mathrm{c}^-\mathrm{c}^+ vz' + \mathrm{b}^-\mathrm{b}^+ xz' \tag{19}$$

$$\mathrm{b}^+\mathrm{d}^- yv' + \mathrm{d}^+ vy' + \mathrm{d}^- xy' + \mathrm{c}^+ zv' = \mathrm{b}^-\mathrm{d}^+ yv' + \mathrm{c}^- zv' + \mathrm{c}^+\mathrm{b}^- vz' + \mathrm{b}^+\mathrm{c}^- xz' \tag{20}$$

where

$$R(\theta \pm \theta') = \begin{pmatrix} 1 & 0 & 0 & \mathrm{d}^\pm \\ 0 & \mathrm{b}^\pm & \mathrm{c}^\pm & 0 \\ 0 & \mathrm{c}^\pm & \mathrm{b}^\pm & 0 \\ \mathrm{d}^\pm & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

and $x' = x(\theta')$, $y' = y(\theta')$, etc.

We start by assuming that one of the elements of K in equation (8) is equal to zero. There will be four cases depending on which element is zero, but only two of these turn out to be different:

(i) $x = 0 = x'$. Using equation (10) we have $v = 0 = v'$ and we are left just with (11) and (9) as independent equations. In order that these two equations be satisfied, we must have that

$$\begin{aligned} z'/y' &= \frac{\mathrm{c}^-\mathrm{d}^+ + \mathrm{b}^- z/y}{\mathrm{b}^- + \mathrm{c}^-\mathrm{d}^+ z/y} \\ &= \frac{\mathrm{c}^+\mathrm{d}^- + \mathrm{b}^+ z/y}{\mathrm{b}^+ + \mathrm{c}^+\mathrm{d}^- z/y} \end{aligned} \tag{22}$$

which implies $(z/y)^2 = 1$. Two solutions are then obtained:

$$K(\theta) = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix} \tag{23}$$

where, from now on, an arbitrary multiplicative function of θ will be omitted.

The case where $v = 0 = v'$ is equivalent to this.

(ii) $z = 0 = z'$. From equation (9), $y = 0 = y'$ and from equations (12) and (10)

$$\begin{aligned} v'/x' &= \frac{\mathrm{c}^+\mathrm{b}^- v/x + \mathrm{c}^-\mathrm{b}^+}{\mathrm{b}^+\mathrm{c}^- v/x + \mathrm{b}^-\mathrm{c}^+} \\ &= \frac{\mathrm{d}^+ v/x + \mathrm{d}^-}{\mathrm{d}^- v/x + \mathrm{d}^+} \end{aligned} \tag{24}$$

which implies $(v/x)^2 = 1$ and then

$$K(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{25}$$

The case where $y = 0 = y'$ is equivalent to this.

We now assume $x(\theta) \neq 0$ and $y(\theta) \neq 0$. Then, equations (11) and (9) imply that

$$z(\theta) = \pm y(\theta) \neq 0 \tag{26}$$

and (12) and (10) require

$$v(\theta) = \pm x(\theta) \neq 0. \tag{27}$$

The matrices $K(\theta)$ in this case have the form

$$K(\theta) = \begin{pmatrix} x(\theta) & \epsilon_1 y(\theta) \\ \epsilon_2 z(\theta) & \epsilon_3 v(\theta) \end{pmatrix} \tag{28}$$

where $\epsilon_1^2, \epsilon_2^2, \epsilon_3^2 = 1$. Omitting an arbitrary multiplicative function of θ, we have only eight different possibilities:

(i) $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$
(ii) $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$
(iii) $\epsilon_1 = \epsilon_3 = 1$ and $\epsilon_2 = -1$
(iv) $\epsilon_2 = \epsilon_3 = 1$ and $\epsilon_1 = -1$
(v) $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon_3 = -1$
(vi) $\epsilon_2 = 1$ and $\epsilon_1 = \epsilon_3 = -1$
(vii) $\epsilon_3 = 1$ and $\epsilon_1 = \epsilon_2 = -1$
(viii) $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$.

Inserting (28) into the rest of the equations, we find only two different equations for $w(\theta) \equiv y(\theta)/x(\theta)$ in all cases. They are

$$w(\theta)/w(\theta') = \frac{\epsilon_3\mathrm{b}^+\mathrm{c}^- + \mathrm{b}^-\mathrm{c}^+ - \epsilon_1\epsilon_2\mathrm{d}^+ - \epsilon_1\epsilon_2\epsilon_3\mathrm{d}^-}{\mathrm{c}^+ + \epsilon_1\epsilon_2\mathrm{b}^+\mathrm{d}^- - \mathrm{c}^- - \epsilon_1\epsilon_2\mathrm{b}^-\mathrm{d}^+} \tag{29}$$

$$w(\theta)/w(\theta') = \frac{\mathrm{b}^-\mathrm{b}^+ + \epsilon_3\mathrm{c}^-\mathrm{c}^+ - \mathrm{d}^-\mathrm{d}^+ - \epsilon_3}{\mathrm{b}^+ + \epsilon_1\epsilon_2\mathrm{c}^+\mathrm{d}^- - \epsilon_3\mathrm{b}^- - \epsilon_1\epsilon_2\epsilon_3\mathrm{c}^-\mathrm{d}^+}. \tag{30}$$

For the previous equations to have a solution, the right-hand side of (29) and (30) must be identical. This can be seen, with some work, to occur for all cases (i)–(viii). One can also see that these expressions factorize as

$$w(\theta)/w(\theta') = \frac{\operatorname{sn}\theta}{[1+\epsilon_1\epsilon_2 k\operatorname{sn}^2\theta]}\Bigg/\frac{\operatorname{sn}\theta'}{[1+\epsilon_1\epsilon_2 k\operatorname{sn}^2\theta']} \tag{31}$$

for the cases where $\epsilon_3 = 1$, and as

$$w(\theta)/w(\theta') = \frac{\operatorname{cn}\theta\operatorname{dn}\theta}{[1+\epsilon_1\epsilon_2 k\operatorname{sn}^2\theta]}\Bigg/\frac{\operatorname{cn}\theta'\operatorname{dn}\theta'}{[1+\epsilon_1\epsilon_2 k\operatorname{sn}^2\theta']} \tag{32}$$

for cases where $\epsilon_3 = -1$. We, therefore, have a θ-independent free parameter in the general solution that we call λ. The solution then reads

$$w(\theta) = \lambda\frac{\operatorname{sn}\theta}{[1+\epsilon_1\epsilon_2 k\operatorname{sn}^2\theta]} \tag{33}$$

when $\epsilon_3 = 1$ and

$$w(\theta) = \lambda\frac{\operatorname{cn}\theta\operatorname{dn}\theta}{[1+\epsilon_1\epsilon_2 k\operatorname{sn}^2\theta]} \tag{34}$$

when $\epsilon_3 = -1$. It can be noticed that these solutions are easily obtained by the residue of (29) when $\theta \to 0$ if $\epsilon_3 = 1$, or the limit $\theta \to 0$ of the same equation when $\epsilon_3 = -1$.

We summarize the general solution of the factorization equations for the eight-vertex model as

$$K_A(\theta) = \begin{pmatrix} [1+\epsilon k\operatorname{sn}^2\theta] & \epsilon\lambda_{HA}\operatorname{sn}\theta \\ \lambda_{HA}\operatorname{sn}\theta & [1+\epsilon k\operatorname{sn}^2\theta] \end{pmatrix} \tag{35}$$

and

$$K_B(\theta) = \begin{pmatrix} [1+\epsilon k\operatorname{sn}^2\theta] & \epsilon\lambda_{HB}\operatorname{cn}\theta\operatorname{dn}\theta \\ \lambda_{HB}\operatorname{cn}\theta\operatorname{dn}\theta & -[1+\epsilon k\operatorname{sn}^2\theta] \end{pmatrix} \tag{36}$$

where $\epsilon^2 = 1$ and $\lambda_{HA}, \lambda_{HB}$ are arbitrary parameters. That is, we find two families of solutions each depending on a continuous and a discrete parameter. (The discrete parameter takes only two values.)

These solutions lead, in the trigonometric limit $k = 0$, to only some specific cases of the general solution for the six-vertex R-matrix discussed in the next section.

We now look for the Hamiltonians obtained by the first derivative of the transfer matrix [3]

$$H = \sum_{n=1}^{N-1} h_{n,n+1} + \frac{1}{2}\left(K_1^-(0)^{-1}\right)\dot{K}_1^-(0) + \frac{\operatorname{tr}_0[K_0^{+t}(-\eta)h_{N0}]}{\operatorname{tr}[K^+(-\eta)]} \tag{37}$$

where

$$h_{n,n+1} = \dot{R}_{n,n+1}(0) \tag{38}$$

and the term $(K_1^-(0)^{-1})\dot{K}_1^-(0)$ generalizes the formula for the Sklyanin Hamiltonian to the case when $K^-(0) \neq 1$. This formula is only defined when $\mathrm{tr}[K^+(-\eta)] \neq 0$ and $\det[K^-(0)] \neq 0$.

We see in equation (36) that, for the second family of solutions, the trace of K is zero. For this second family, we will then not have a well defined Hamiltonian from the first derivative of the transfer matrix.

When

$$\mathrm{tr}[K^+(-\eta)] = 0 \tag{39}$$

and

$$\mathrm{tr}_0[K_0^{+\prime}(-\eta)h_{N0}] \propto 1 \tag{40}$$

a well defined Hamiltonian with only nearest-neighbour interactions is obtained from the second derivative of the transfer matrix as shown in [10]. However, for the present second family of solutions, condition (40) does not hold. Furthermore $\dot{K}_B(0) = 0$, which gives only a trivial boundary term at the left end. The same happens with solutions (23) and (25) where one of the elements is zero.

If condition (39) holds, but not equation (40), one obtains, from the second derivative of the transfer matrix, a Hamiltonian with terms that couple every pair of sites in the bulk with the boundary, i.e. a non-local Hamiltonian arises.

The Hamiltonians associated to the first family of solutions (35) are given by

$$H = \sum_{i=1}^{N-1} h_{n,n+1}^{XYZ} + \xi_-\sigma_1^\alpha + \xi_+\sigma_N^\beta. \tag{41}$$

Here, α and β can take the values x or y in all possible combinations and the $\xi_\pm$ are arbitrary parameters proportional to λ_{HA}.

As is clear by rotating the axis, we can make the indices α and β in equation (41) also take the value z.

Equation (41) gives the most general choice of BC compatible with integrability for the XYZ chain besides periodic and twisted BC. By twisted BC, we mean

$$\sigma_{N+1}^\alpha = M\sigma_1^\alpha M^{-1} \tag{42}$$

where $\alpha = x, y, z$ and the twisting matrix M stands for a discrete symmetry of the eight-vertex model. That is, $M = \sigma^z$ or σ^x.

In conclusion, the XYZ Hamiltonian is integrable with BC that correspond to the coupling with a magnetic field on the end sites oriented along parallel or orthogonal directions.

3. General K-matrices for the six-vertex model (*XXZ* and *XXX* chains)

In this section, we briefly review the results of [4] concerning the general solution for the K-matrices of the XXZ chain and give the general solution for the XXX case.

The R-matrix of the six-vertex model is given by

$$R(\theta)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \dfrac{\sin\gamma}{\sin(\theta+\gamma)} & \dfrac{\sin\theta}{\sin(\theta+\gamma)} & 0\\ 0 & \dfrac{\sin\theta}{\sin(\theta+\gamma)} & \dfrac{\sin\gamma}{\sin(\theta+\gamma)} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{43}$$

and the general solution to the factorization equations in this model is given by [4]

$$K(\theta,\beta,\lambda,\mu,\xi)=\begin{pmatrix}\beta\sin(\xi+\theta) & \mu\sin 2\theta\\ \lambda\sin 2\theta & \beta\sin(\xi-\theta)\end{pmatrix} \tag{44}$$

where β, ξ, μ and λ are arbitrary parameters. The associated Hamiltonians to this K-matrix follow from the procedure used above. Defining

$$K^{\pm}(\theta)=K(\theta,\beta_\pm,\lambda_\pm,\mu_\pm,\xi_\pm) \tag{45}$$

the following Hamiltonians are obtained:

$$\begin{aligned}H=&\sum_{n=1}^{N-1}(\sigma_n^x\sigma_{n+1}^x+\sigma_n^y\sigma_{n+1}^y+\cosh\gamma\,\sigma_n^z\sigma_{n+1}^z)\\ &+\sin\gamma(b_-\sigma_1^z-b_+\sigma_N^z+c_-\sigma_1^- -c_+\sigma_N^- +d_-\sigma_1^+ -d_+\sigma_N^+)\end{aligned}$$

where the parameters $b_\pm$, $c_\pm$ and $d_\pm$ follow from $\lambda_\pm$, $\mu_\pm$, $\xi_\pm$ and $\beta_\pm$ as shown:

$$b_\pm=\cot\xi_\pm \qquad c_\pm=\frac{2\lambda_\pm}{\beta_\pm\sin\xi_\pm} \qquad d_\pm=\frac{2\mu_\pm}{\beta_\pm\sin\xi_\pm}. \tag{47}$$

Here $\beta_\pm, \xi_\pm \neq 0$ so that $\det[K^-(0)]\neq 0$ and $\mathrm{tr}[K^+(-\eta)]\neq 0$.

Equation (46) gives the most general choice of boundary terms compatible with integrability for the XXZ chain besides periodic and twisted BC. In the present case, one can twist the BC as

$$\sigma_{N+1}^\alpha=M\sigma_1^\alpha M^{-1} \tag{48}$$

where $M=\sigma^x$ or $M=e^{i\omega\sigma_z}$, $0<\omega<2\pi$.

Looking to equations (10) and (12), it is now possible to see why in the elliptic case we lose a continuous parameter that appears in the trigonometric limit. In the trigonometric case, $d^+=d^-=0$ and we have only the constraint of equation (12), which gives a continuous family of solutions. The same happens with equations (9) and (11), losing again a continuous parameter from the elliptic case.

It is also interesting to see which Hamiltonians are obtained from the trigonometric limit of the K-matrices obtained in the preceding section. When $k=0$ in (35), one obtains

$$K_{TA}(\theta)=\begin{pmatrix}1 & \epsilon\lambda_{HA}\sin\theta\\ \lambda_{HA}\sinh\theta & 1\end{pmatrix} \tag{49}$$

and this is seen to correspond to solution (44) with $\beta=1$, $\xi=\pm\pi/2$ and $\mu,\lambda=\pm\lambda_{HA}/2$. The corresponding Hamiltonians are obtained from the substitution of these values of the parameters in equations (45)–(47).

For solution (36), when $k = 0$, one obtains

$$K_{TB}(\theta) = \begin{pmatrix} 1 & \epsilon\lambda_{HB}\cos\theta \\ \lambda_{HB}\cos\theta & -1 \end{pmatrix} \tag{50}$$

which corresponds to solution (44) with $\beta = 1, \xi = 0, \pi$ and $\mu, \lambda = \pm\lambda_{HB}/2$. As discussed in the previous section, this limit leads to a Hamiltonian which includes a non-local coupling with the boundaries.

It is interesting to note at this point that the trigonometric limit of the K-matrices for the eight-vertex model does not lead to an $SU_q(2)$-invariant Hamiltonian. This is not the case for the free-fermion eight-vertex model where the $CH_q(2)$ symmetry given by the elliptic K-matrices 'contracts' to a $U_q(gl(1,1))$ symmetry in the trigonometric limit [10, 12].

Let us now look for the general solution of the factorization equations in the rational limit of the R-matrix (43) given by

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1}{(\theta+1)} & \dfrac{\theta}{(\theta+1)} & 0 \\ 0 & \dfrac{\theta}{(\theta+1)} & \dfrac{1}{(\theta+1)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{51}$$

The equations for the K-matrix when R is rational follow by substituting the sine functions by their arguments (that is, $\sin(\omega)$ by ω) in all the equations. The number of independent equations is the same in the rational and trigonometric cases. (This number decreases going from the elliptic to the trigonometric case.) Thus, the general solution is

$$K(\theta, \beta, \lambda, \mu, \xi) = \begin{pmatrix} \beta(\xi+\theta) & \mu\theta \\ \lambda\theta & \beta(\xi-\theta) \end{pmatrix} \tag{52}$$

and using equation (37), one obtains

$$H = \sum_{n=1}^{N-1}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z) + b_-\sigma_1^z - b_+\sigma_N^z + c_-\sigma_1^- - c_+\sigma_N^- + d_-\sigma_1^+ - d_+\sigma_N^+ \tag{53}$$

where we have scaled by a factor of 2γ and omitted a term proportional to the identity operator. In this case, the parameters $b_\pm$, $c_\pm$ and $d_\pm$ follow from $\lambda_\pm$, $\mu_\pm$, $\xi_\pm$ and $\beta_\pm$ as

$$b_\pm = \frac{1}{\xi_\pm} \qquad c_\pm = \frac{\lambda_\pm}{\beta_\pm\xi_\pm} \qquad d_\pm = \frac{\mu_\pm}{\beta_\pm\xi_\pm} \tag{54}$$

where $\beta_\pm, \xi_\pm \neq 0$ so that $\det[K^-(0)] \neq 0$ and $\mathrm{tr}[K^+(-\eta)] \neq 0$.

This equation again provides the most general choice of boundary terms compatible with integrability for the XXX chain besides periodic and twisted BC.

4. Conclusions

We have presented the general solution to the surface factorization equations for the XYZ, XXZ and XXX models providing in this way the most general boundary terms compatible with integrability. One can expect that if any kind of quantum group invariance is possible in the XYZ chain, the necessary boundary terms will be provided by those of Hamiltonian (41). For the XYZ chain a geneneralization of the construction for the eigenvalues and eigenvectors of the periodic chain remains to be performed. As the Hamiltonians obtained for the XXZ and XXX models do not commute with J_z, a generalization of the functional Bethe ansatz proposed by Sklyanin [13] for open BC should be useful to find the eigenvalues.

In the context of two-dimensional integrable quantum field theories with boundaries, it is interesting to solve the boundary bootstrap and cross-unitarity equations for these solutions.

Acknowledgments

AGR would like to thank the LPTHE for the kind hospitality and the Spanish MEC for financial support under grant AP90 02620085.

References

[1] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)
[2] de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371
[3] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375
Cherednik I 1984 *Theor. Math. Phys.* **61** 35
[4] de Vega H J and González-Ruiz A 1993 *J. Phys. A: Math. Gen.* **26** L519
[5] Mezincescu L and Nepomechie R 1991 *J. Phys. A: Math. Gen.* **24** L17–23; 1991 *Int. J. Mod. Phys.* A **6** 5231; 1992 *Int. J. Mod. Phys.* A **7** 5657
[6] Fring A and Köberle R 1993 Factorized scattering in the presence of reflecting boundaries *Preprint* USP-IFPSC/TH/93-06
[7] Ghoshal S and Zamolodchikov Al 1993 Boundary S-matrices and boundary state in two-dimensional integrable quantum field theory *Preprint* RU-93-20
[8] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523–56
[9] Destri C and de Vega H J 1992 *Nucl. Phys.* B **374** 692–719; **385** 361–91
[10] Cuerno R and González-Ruiz A 1993 *J. Phys. A: Math. Gen.* **26** L605
[11] Uglov D B 1993 The Lie algebra of the $sl(2, C)$–valued automorphic finctions on a torus, and the quantum bialgebra associated with the eight-vertex R-matrix *Preprint* SUNY
[12] Cuerno R, Gómez C, López E and Sierra G 1993 *Phys. Lett.* **307B** 56
[13] Sklyanin E K *Nankai Lectures*